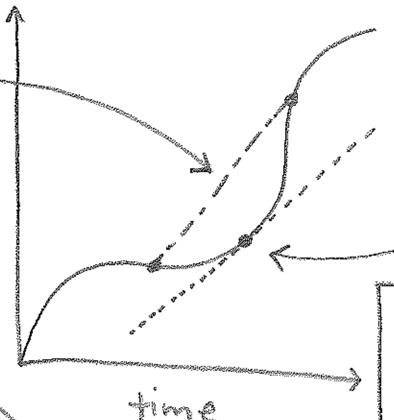
Calc-based Mechanics

Pencil Notes

$\Delta x = x_2 - x_1$ DISPLACEMENT - how far something's gone

$V_{avg} = \frac{\Delta x}{\Delta t}$ AVERAGE VELOCITY — slope of the secant line

$V_{inst} = \frac{dx}{dt}$ INSTANTANEOUS VELOCITY — slope of the tangent line



SPEED- the magnitude of velocity

speed = |velocity|

VELOCITY- how fast it's going

y = position

y' = velocity

y'' = acceleration

ACCELERATION- how fast it's speeding up/slowing down

$a_{avg} = \frac{\Delta v}{\Delta t}$ AVERAGE ACCELERATION $a_{inst} = \frac{dv}{dt}$ INSTANTANEOUS ACCELERATION

(we're assuming you know calculus and how to differentiate and integrate already. If ya don't, learn.)

Tools of the CONSTANT ACCELERATION

$x = x_0 + v_0 t + \frac{1}{2} a t^2$

$v = v_0 + at$

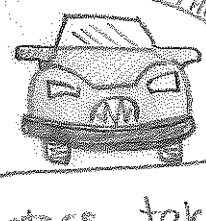
$x - x_0 = \frac{1}{2}(v_0 + v)t$

$t = \frac{v - v_0}{a}$

$v^2 = v_0^2 + 2a(x - x_0)$

$g = 9.8\ m/s^2$ CRASH

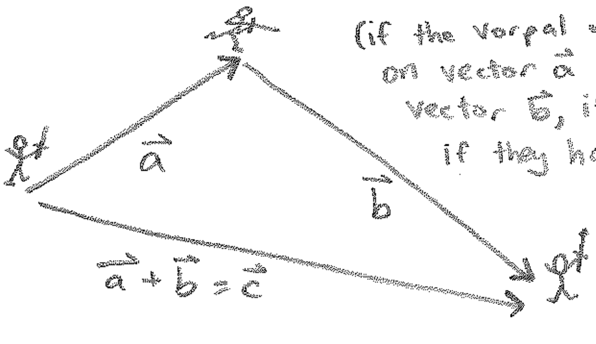
[insert sinister accordion music]



Vectors: physical quantities with magnitude and direction.

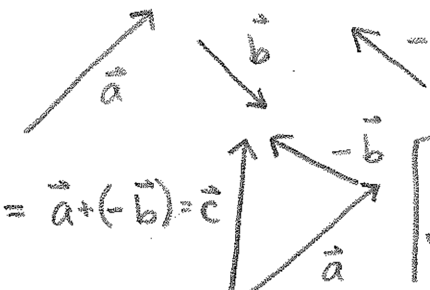
They also make good weapons.*

To add vectors, just draw them together and place the tail of the 2nd vector at the head of the 1st.



(if the vorpal warriors** run on vector $\vec{a}$ and then on vector $\vec{b}$, it's the same as if they had run on $\vec{a} + \vec{b}$, or vector $\vec{c}$.)

To subtract vectors, take the vector you're subtracting from and flip it around.

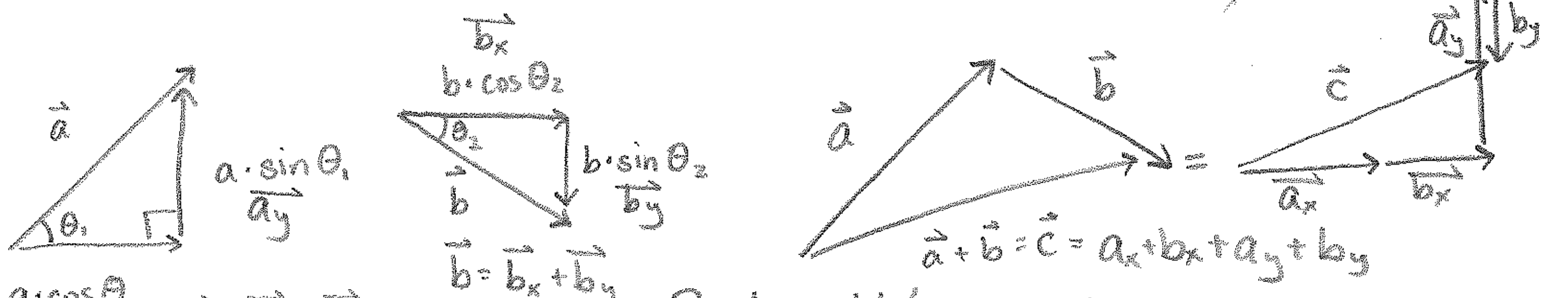


$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \vec{c}$

** Also from the Vortext. From L. Carroll's "Jabberwocky."

* In The Vortext, a story written in fall 2001, The Froggish One (Katy Dieber, '02) wielded a vector against the forces of Ma'ath. It spun around her finger, and, when properly wielded, could eliminate both Ma'ath and Geom's vector (by being placed perpendicular to it.)

You can resolve vectors into their components. This makes it easier to add and subtract them.



$\vec{a} = \vec{a_x} + \vec{a_y}$

$\vec{b} = \vec{b_x} + \vec{b_y}$

$\vec{a} + \vec{b} = \vec{c} = a_x + b_x + a_y + b_y$

So to add/subtract vectors, you can split 'em up into x and y vectors and add/subtract those. Use Trig. And Geometry.

$\vec{a} = a_x + a_y$

$\vec{a_x} = a_x \hat{i}$

$\vec{a_y} = a_y \hat{j}$

$\vec{a_z} = a_z \hat{k}$

there's also this way of writing it.

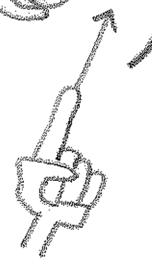
vectors – magnitude + direction

scalars – just magnitude

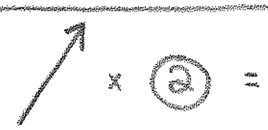
vector · scalar = vector with same direction and magnitude equal to the product of the scalar and the original vector's magnitude.

We'll let Katarina* demonstrate.

I've got a vector and a scalar,** see?

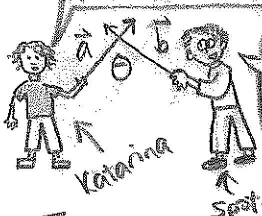


I multiply them together...



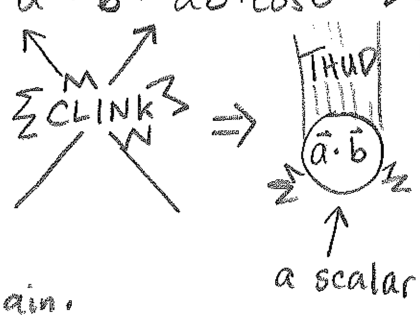
And now I have a vector pointing in the same direction, only twice as big. Nifty.

Now Scott-o-more*** and I will show you how to multiply vectors.

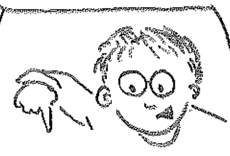


First the scalar product. Its product is a scalar. Big surprise.

$\vec{a} \cdot \vec{b} = ab \cdot \cos\theta$ = DOT PRODUCT



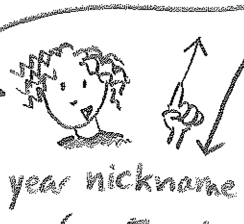
This is the same as having one vector multiplied by the scalar of the other vector that's in the same direction of the first vector.

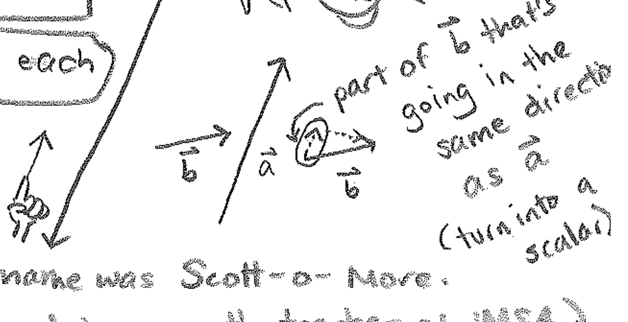


This is important, so we'll write it again.

DOT PRODUCT = $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = ab \cdot \cos\theta$

The dot product is also the sum of the products of each of the component vectors. This comes in handy when you need to find the angle between two vectors.





*** Written by Scott "Spot" Stiefel, '04. His sophomore year nickname was Scott-o-More.

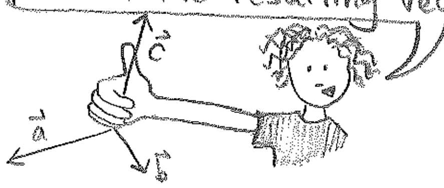
** Scalars provided courtesy of Fogel-Buddha (Dr. Fogel is a math teacher at IMSA.)

* The aforementioned "Froggish One."

CROSS PRODUCT $= ab \cdot \sin \phi = (a_x + a_y + a_z) \times (b_x + b_y + b_z)$ = another vector

(φ ← smaller angle)

This'll give you a vector perpendicular to both $\vec{a}$ and $\vec{b}$. Use your right hand to find out which way; sweep your fingers from $\vec{a}$ to $\vec{b}$. Your thumb points in the resulting vector's direction.



Easy memory trick; add all the down-left products, then subtract all the up-right products.

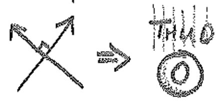
$$\begin{array}{ccc|c} \hat{i} & \hat{j} & \hat{k} & \\ a_x & a_y & a_z & a_x \\ b_x & b_y & b_z & b_x \end{array}$$

(↗ −, ↘ +)

$(a_y b_z - b_y a_z)\,\hat{i} + (a_z b_x - b_z a_x)\,\hat{j} + (a_x b_y - b_y a_x)\,\hat{k}$ = Cross product

For each $\hat{i}\hat{j}\hat{k}$ term, you use the two other directions.

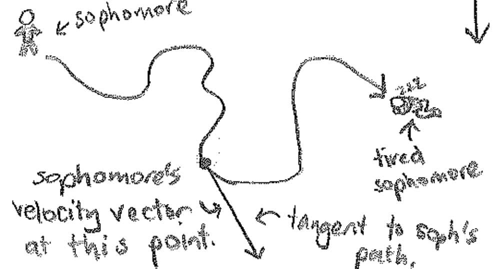
Note that when $\theta = 90$, dot product $= 0$.*



Also, when $\phi = 0°$ or $180°$;

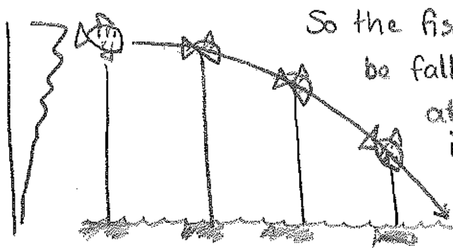
the cross product $= 0$.

The direction of motion on a 2d plane is the tangent line to the path.



PROJECTILE MOTION

Vertical and horizontal movements are independent of each other.



So the fish** may be falling down at an increasing speed, but its shadow on the water will still go at the same rate.

PATH $= y = (\tan \theta_0)x - \dfrac{gx^2}{2(v_0 \cdot \cos\theta_0)^2}$

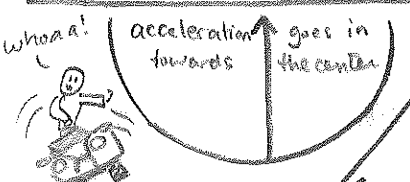
RANGE $= \dfrac{V_0^2}{g} \cdot \sin(2\theta)$

Without air resistance, maximum range is at 45°.

CIRCULAR MOTION

$a = \dfrac{v^2}{r}$ = CENTRIPETAL ACCEL.

T = period of revolution $= \dfrac{2\pi r}{v}$

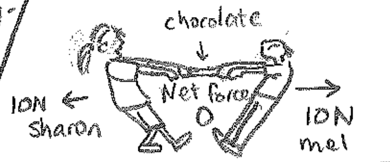


Newton's 1st Law: If no net force acts on a body, then the body's velocity cannot change.

Newton's 2nd Law: Force = Mass · Accel.

(note: Mass ≠ weight. Weight = mass · gravity.

Forces superposition themselves.

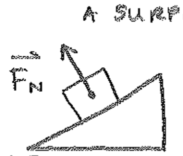


NEWTON'S 3rd LAW: Action-reaction: Mel and Sharon pull equally hard on one another.

FORCES

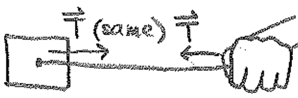
NORMAL FORCE

FOUND: PERPENDICULAR TO A SURFACE



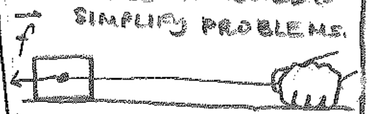
TENSION

FOUND: IN TAUT ROPES OR STRINGS, AWAY FROM THE OBJECT



FRICTION

FOUND: AWAY FROM THE DIRECTION OF MOTION. MAY BE NEGLECTED TO SIMPLIFY PROBLEMS.



$\vec{f}_s$ = static frictional force

$\vec{f}_k$ = kinetic frictional force

$\vec{f}_k < \vec{f}_s$ (usually)

$f_{s\,max} = \mu_s \cdot N$ = biggest force you can apply before the object moves

μ = mu = coefficient of friction

$f_k = \mu_k \cdot N$

Frictional forces oppose the direction of motion.

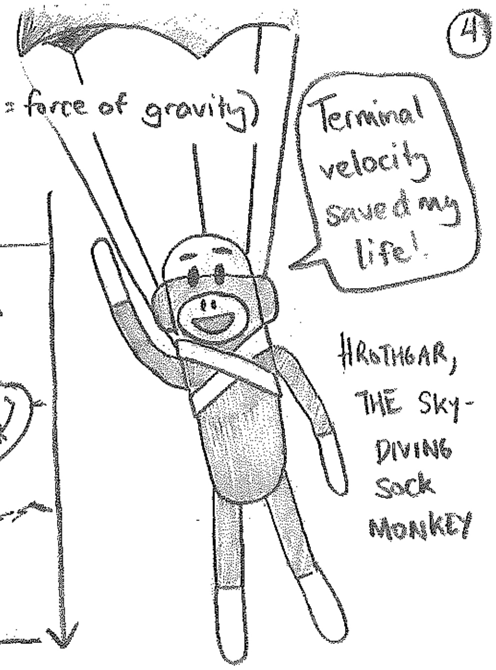
** Vorpal Fish, green; owned by Erik Volkman, '03 (Satchel Boy).

* This is the only way to eliminate the vector of Geom and defeat the Ma'ath.

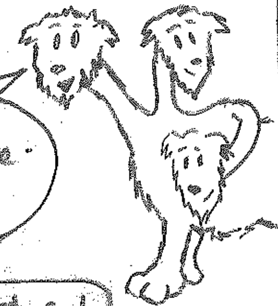
C = drag coefficient. No, we don't mean British comedians in dresses.

TERMINAL VELOCITY = $v_t = \sqrt{\frac{2F_g}{C\rho A}}$ (F_g = force of gravity)

$D_{rag\ force} = \frac{1}{2} C \rho A v^2$ (ρ = fluid density, A = cross-sectional area)



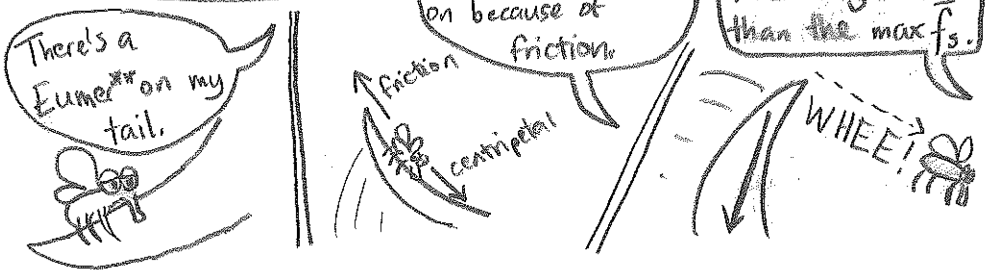
Let's return to uniform circular motion. Fluffy?*



Centripetal acceleration pulls things inwards. If you don't have enough friction - or some other force - to counteract it, you'll go flying off the curve. Observe.

$a = \frac{v^2}{R}$

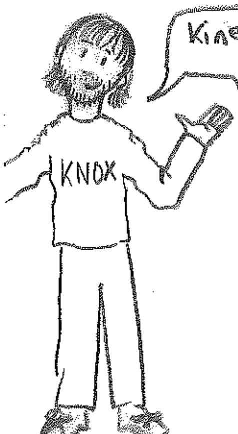
$F_{centripetal} = m\frac{v^2}{R}$



Centripetal forces accelerate things by changing their direction, not their speed.

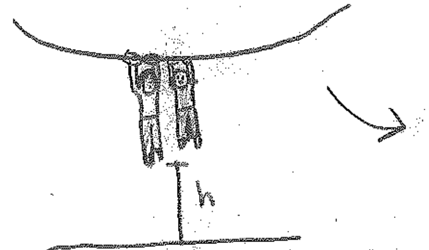
(If, on the other hand, you have no air resistance, a bullet fired at the same instant the monkey falls will hit the monkey if the gun is initially aimed at it, since they'll fall at the same rate.)

Time to talk about Energy. Here's Thor of Knoxxx.***



Kinetic energy is when an object is moving. Potential energy is when something could turn into kinetic energy.

For example, look at Mel and Katarina.



They're hanging from a vine. They're not moving. They have potential energy.

Potential Energy = m · g · h

Whoop. Now they have Kinetic and potential.

$KE = \frac{1}{2}mv^2$

Energy is measured in kilogram-meters-squared per second squared. The easy way to pronounce this is "Joules." (or J.)

Now it's all Kinetic at the bottom of their swing.

NOW WE GO TO WORK.

Work? Gah. Man, I'm outta here.

Work, n. 1. Energy transferred to/from an object by means of a force on it.
2. The bane of students.
3. Substitute for sleep.

ENERGY → OBJECT = + WORK
ENERGY ← OBJECT = − WORK

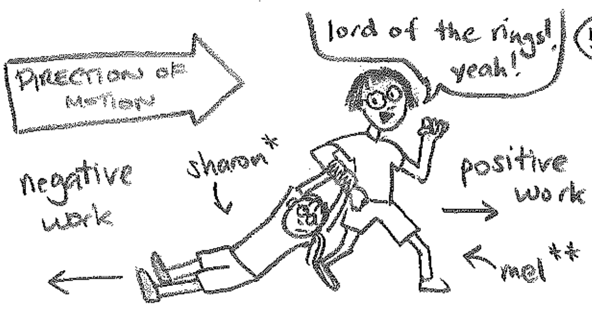
*** Matt "Thor" Femrite ('01) of Knox College, an Aloorn (departed spirit of the Eemsa bubble.)

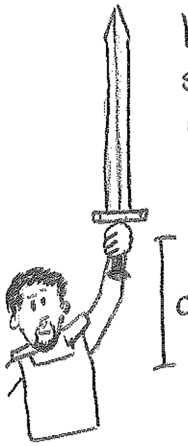
** Symbiont bugs with the Zar Chasms. They congregate in the Pit of Ack during Sudo Festivals.

* Sharon David ('03)'s Vortext pet. Fluffy is a Zar Chasm, and grows when cynical remarks are utter

WORK $= \vec{F} d \cdot \cos\phi = \vec{F} \cdot \vec{d}$ (constant force)

Forces do positive work if they go in the direction of displacement, and negative if they go the opposite way. $\Delta K = W$ (change in kinetic energy equals the work done.)



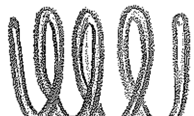


In lifting his sword, Danjoe of Langan*** does work on it. So does gravity.

d

$W_{gravity} = m \cdot g \cdot d \cdot \cos\phi$

(gravity, unlike IMSA students, is ~~always~~ diligent.)



SPRING FORCES

$F = -kx$ ← Hooke's Law

k = spring constant
Stretched spring = +x
Compressed spring = −x

$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$ (work done by a spring)

$$W = \int_{x_i}^{x_f} F(x)\,dx$$

WORK DONE BY A GENERAL VARIABLE FORCE

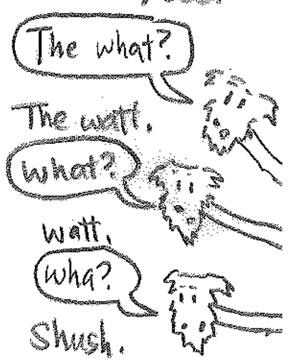
POWER = work per unit time.

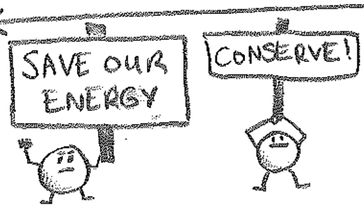
$P_{avg} = \frac{W}{\Delta t}$

$P_{inst} = \frac{dW}{dt}$

$P = \vec{F} \cdot \vec{v}$

The unit of power, the watt, is one Joule/sec.





Energy is conserved.
The change in potential energy

$\Delta U = -\int_{x_i}^{x_f} F(x)\,dx$

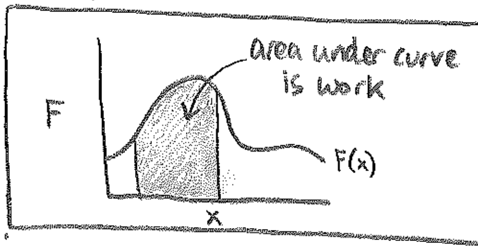
equals the work done to change a system's configuration.

$U_y = mgy$ − gravitational

$U(x) = \frac{1}{2}kx^2$ − elastic

CONSERVATION OF POTENTIAL ENERGY

Mechanical Energy = K + U
(sum of potential and kinetic)



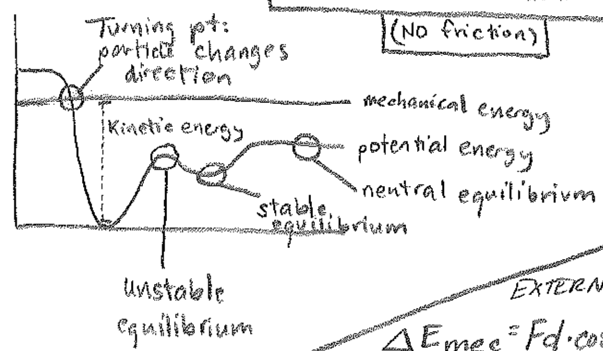
$\Delta E_{thermal} = f_k d$ (friction sliding)

$W = \Delta E_{thermal} + \Delta E_{mechanical}$ (with friction)

The total energy E of an isolated system cannot change.

PE Curves

$Work = \Delta E_{mechanical}$ (NO friction)



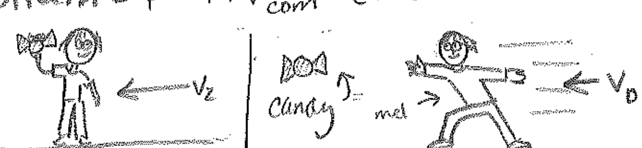
EXTERNAL

$\Delta E_{mec} = Fd \cdot \cos\phi$

changing

Linear Momentum $= \vec{P} = M\vec{V}_{com}$ (conserved)

$v_f - v_i = v \ln \frac{m_i}{m_f}$ varying mass



$(M_{candy} + M_{mel})\, v_2 = m_{mel}\, v_0$

CENTER OF MASS

$\frac{1}{M}\sum_{i=1}^{n} m_i r_i$ particles

$\frac{1}{M}\int x\,dm$ continuous

$\frac{1}{V}\int x\,dV$ uniform ρ

$\rho = \frac{M}{V} = \frac{dm}{dV}$

IMPULSE

$\vec{J} = \int_{t_i}^{t_f} F(t)\,dt$

$\vec{J} = F_{avg}\Delta t$

$\vec{J} = m\Delta v$

1 dimensional elastic

$V_1 = \frac{m_1 - m_2}{m_1 + m_2} V_0$ $V_2 = \frac{2m_1}{m_1 + m_2}$ $(V_1) \rightarrow (V_2)$

*** Joseph Daniel "Danjoe" Langan ('02). The original Keeper of the Lore.
** Class of '03, Sharon's roommate. Overly obsessed with massive LOTR swordage.
* Class of '03. Didn't see Fellowship until Two Towers came out.